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ADDENDUM

New conjecture for the $SU_q(N)$ Perk–Schultz models

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Abstract
We present a new conjecture for the $SU_q(N)$ Perk–Schultz models. This conjecture extends a conjecture presented in our article (Alcaraz F C and Stroganov Yu G *J. Phys. A: Math. Gen.* **35** 6767–87).

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In [1] (to which we refer hereafter as I), based on numerical evidence, we present a series of conjectures about the eigenspectra of the $SU_q(N)$ invariant Perk–Schultz Hamiltonians, given by I.1 with $p = 0$. Subsequent extensive numerical work indicates that the conjecture (3) of I (I.85) is just a particular case of a more general one, relating the ground-state energy of different $SU_q(N)$ quantum chains. This new conjecture can be stated as follows:

Conjecture. The difference of the ground-state energy of the $SU_q(N)$ and $SU_{q'}(K)$ model in an open chain of length L is given by

$$E_0[SU_q(N), q = \exp(\mathrm{i}\pi K/(N+K))] - E_0[SU_q(K), q = \exp(\mathrm{i}\pi N/(N+K))]$$
$$= 2(1-L)\sin\left(\frac{\pi(N-K)}{2(N+K)}\right), \tag{1}$$

where $N \neq K = 1, 2, \ldots$ and $E_0[SU_q(1), q)] = 0$. The particular case $N > 1$ and $K = 1$ gives

$$E_0[SU_q(N), q = \exp(\mathrm{i}\pi/(N+1))] = 2(1-L)\sin\left(\frac{\pi(N-1)}{2(N+1)}\right), \tag{2}$$

recovering the earlier announced conjecture (I.85). The results (1) and (2) give exact finite-size corrections supporting earlier conjectures about the operator content of the Perk–Schultz models. As is well known [2] as a consequence of conformal invariance the finite-size corrections of the ground-state energy of critical chains with free boundary conditions are given by

$$E_0(L)/L = e_\infty + f_\infty/L - \frac{\pi c}{24L^2} + o(L^{-2}), \tag{3}$$

where e_∞ (f_∞) is the ground-state energy per site (surface energy) at the bulk limit $L \to \infty$ and c is the conformal anomaly of the effective underlying conformal field theory defined on a semi-infinite plane. Relations (3) and (2) imply that in the case of the $SU_q(N)$ model with $q = \exp(\mathrm{i}\pi N/(N+1))$ the conformal anomaly has the value $c = 0$ for all $N \geqslant 2$ and

$$e_\infty = -f_\infty = -2\sin\left(\frac{\pi(N-1)}{2(N+1)}\right). \tag{4}$$

Moreover, all the other finite-size corrections appearing in (3) are identically zero! This result can be understood from the conjectured operator content of the model. The conformal dimensions of the $SU_q(N)$ model are expected to be given by a generalized Coulomb gas description

$$x(\vec{n}, \vec{m}) = \frac{x_p}{2}\sum_{i,j=1}^{N-1} n_i C_{ij} n_j + \frac{1}{8x_p}\sum_{i,j=1}^{N-1} m_j (C^{-1})_{i,j} m_j, \tag{5}$$

where C is the $SU(N)$ Cartan matrix, and

$$x_p = \frac{\pi - \gamma}{2\pi}, \qquad q = \mathrm{e}^{\mathrm{i}\gamma}. \tag{6}$$

The vectors $\vec{n} = (n_1, \ldots, n_N)$ and $\vec{m} = (m_1, \ldots, m_n)$ label the possible values of the electric and magnetic charges in the Coulomb gas representation. The possible values of $\vec{n}$ and $\vec{m}$ depend on the parity of the lattice and the boundary conditions where the quantum chain is defined. For the present case of free boundaries the conformal anomaly of the effective theory is conjectured to be given by [3]

$$c = (N-1) - 12x(\vec{0}; m_1, m_2, \ldots, m_N) \tag{7}$$

with

$$m_1 = m_2 = \cdots = m_N = 2\frac{\gamma}{\pi}. \tag{8}$$

In the case of relation (2) we should use $\gamma = \pi/(N+1)$ so that

$$x_p = \frac{N}{2(N+1)}, \qquad \text{and} \qquad x(\vec{0}; 2\gamma/\pi, \ldots, 2\gamma/\pi) = (N-1)/12, \tag{9}$$

and from (7) we obtain $c = 0$ for all N, explaining the previous result.

Expressions (1) and (3) for arbitrary values of N and K also imply the interesting relations among the ground-state energy and surface energy in the bulk limit:

$$e_\infty\left(SU_q(N), \gamma = \frac{\pi K}{N+K}\right) - e_\infty\left(SU_q(K), \gamma = \frac{\pi N}{N+K}\right) = -2\sin\left(\frac{\pi(N-K)}{2(N+K)}\right),$$

$$f_\infty\left(SU_q(N), \gamma = \frac{\pi K}{N+K}\right) - f_\infty\left(SU_q(K), \gamma = \frac{\pi N}{N+K}\right) = 2\sin\left(\frac{\pi(N-K)}{2(N+K)}\right).$$

These relations can also be verified from the exact expressions e_∞ and f_∞ derived in the bulk limit $L \to \infty$ [4]. Moreover, (1) and (3) relate the conformal anomalies of distinct effective conformal field theories

$$c\left[S_qU(N), \gamma = \frac{\pi K}{N+K}\right] = c\left[S_qU(K), \gamma = \frac{\pi N}{N+K}\right]. \tag{10}$$

In fact this equality supports the conjectures (7) and (8) since both sides of the last equation give the effective conformal anomaly

$$c = \frac{N^2 + K^2 + NK - N^2K^2 - K - N}{24(N+K)}, \tag{11}$$

for the related quantum chains.

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